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Bäcklund–Darboux transformations for the coupled KP hierarchy

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Abstract
Using the geometry of the infinite isotropic Grassmannian related to the Hirota–Ohta-coupled KP hierarchy, we construct its elementary Bäcklund–Darboux transformations.

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1. Introduction

The Hirota–Ohta-coupled KP system was introduced in [11] to obtain a Hirota bi-linear hierarchy of KP type that has Pfaffian solutions (see also [7–9, 21] for more hierarchies of this type). It contains the following Hirota bilinear equations:

$$
\begin{aligned}
&\left(D_1^4 - 4D_1D_3 + 3D_2^2\right)\tau\cdot\tau = 24\overline{\sigma}\sigma\\
&\left(\left(D_1^3 + 2D_3\right)D_2 - 3D_1D_4\right)\tau\cdot\tau = 12D_1\overline{\sigma}\cdot\sigma\\
&\left(D_1^6 + 40D_1^3D_3 + 40D_3^2 - 216D_1D_5 + 45D_1^2D_2^2 + 90D_2D_4\right)\tau\cdot\tau\\
&\qquad\qquad = 360\left(D_1^2 + D_2\right)\overline{\sigma}\cdot\sigma\\
&\left(D_1^6 - 20D_1^3D_3 - 80D_3^2 + 144D_1D_5 - 45D_1^2D_2^2\right)\tau\cdot\tau = -360D_1^2\overline{\sigma}\cdot\sigma\\
&\ldots
\end{aligned}
\tag{1.1}
$$

$$
\begin{aligned}
&\left(D_1^3 + 2D_3 + 3D_1D_2\right)\sigma\cdot\tau = 0\\
&\left(D_1^4 - 4D_1D_3 - 3D_2^2 - 6D_4\right)\sigma\cdot\tau = 0\\
&\left(3D_1^5 + 72D_5 + \left(-10D_1^3 + 40D_3\right)D_2 - 15D_1D_2^2 + 30D_1D_4\right)\sigma\cdot\tau = 0\\
&\left(D_1^5 - 10D_1^2D_3 + 24D_5 + \left(-5D_1^3 + 20D_3\right)D_2\right)\sigma\cdot\tau = 0\\
&\ldots
\end{aligned}
\tag{1.2}
$$

$$
\begin{aligned}
&\big(D_1^3+2D_3-3D_1D_2\big)\overline{\sigma}\cdot\tau=0\\
&\big(D_1^4-4D_1D_3-3D_2^2+6D_4\big)\overline{\sigma}\cdot\tau=0\\
&\big(3D_1^5+72D_5+\big(10D_1^3-40D_3\big)D_2-15D_1D_2^2-30D_1D_4\big)\overline{\sigma}\cdot\tau=0\\
&\big(D_1^5-10D_1^2D_3+24D_5+\big(5D_1^3-20D_3\big)D_2\big)\overline{\sigma}\cdot\tau=0\\
&\cdots
\end{aligned}
\tag{1.3}
$$

where

$$
P(D)f\cdot g=P\left(\frac{\partial}{\partial y_1},\frac{\partial}{\partial y_2},\ldots\right)(f(x+y)g(x-y)|_{y_1=y_2=\cdots=0}.
$$

This hierarchy of equations has gone through repeated discoveries and re-discoveries, e.g. it is the same as the Pfaff lattice of Adler and van Moerbeke *et al* [1–5], which plays a role in the theory of random matrices. It has symmetric and symplectic matrix integrals as solutions (see also [18, 20]). It is also the same as one of the DKP hierarchies of Kac and the author [17], which describes some infinite-dimensional Clifford group orbit (see also [20]). The original work of Hirota and Ohta [11] from 1991 is only very recently broadly cited. However, the spin-representation of this hierarchy, together with its bilinear formulation (and that of the modified hierarchy, as mentioned in theorem 2.1 of this paper) already appears in 1983 in a paper by Jimbo and Miwa [14]. A wavefunction formulation can already be found in [12], which dates from 1989. Using this spin-representation formulation we will construct in a geometric way elementary Bäcklund–Darboux transformations in the style of [10]. To do that we will first formulate the hierarchy in the framework of [14, 17] and show that the above equations indeed appear in this hierarchy. Sections 2, 3, except the part on eigenfunctions, also appear in some form in [17]. The wavefunction, however has some slightly different form there. Although proposition 4.1, which is the key to the elementary Bäcklund–Darboux transformation, also appears in [17], it was not yet used to construct these transformations. Recently, there has been considerable interest in particular solutions of this hierarchy (see, e.g., [13] or [6]) and Bäcklund–Darboux-type transformations could prove to be especially useful in this context.

2. Clifford algebra construction of the coupled KP

The Clifford algebra approach or spin-representation formulation of the coupled KP hierarchy first appeared in [14] (see also [17]). The approach we present here is based on the well-known construction of vertex algebras (see, e.g., [15]).

Let F be the vector space $F=\mathbb{C}[q,q^{-1},t_1,t_2,\ldots]$. We decompose F as follows:

$$
F=\bigoplus_{k\in\mathbb{Z}}F_k\qquad\text{where}\qquad F_k=q^k\mathbb{C}[t_1,t_2,\ldots].
$$

Consider the vertex operators, i.e., generating series of operators

$$
\begin{aligned}
\psi^{\pm}(z)&=\sum_{i\in\frac12+\mathbb{Z}}\psi_i^{\pm}z^{-i-\frac12}\\
&=q^{\pm1}z^{\pm q\frac{\partial}{\partial q}}\,\mathrm{e}^{\pm\xi(t,z)}\,\mathrm{e}^{\mp\eta(t,z)}
\end{aligned}
\tag{2.1}
$$

where

$$
\xi(t,z)=\sum_{k=1}^{\infty}t_kz^k\qquad\text{and}\qquad\eta(t,z)=\sum_{k=1}^{\infty}\frac1k\frac{\partial}{\partial t_k}z^{-k}.
$$

Then (see, e.g., [15])

$$\psi_i^{\pm}\psi_j^{\pm}+\psi_j^{\pm}\psi_i^{\pm}=0 \qquad \psi_i^{\pm}\psi_j^{\mp}+\psi_j^{\mp}\psi_i^{\pm}=\delta_{i,-j} \tag{2.2}$$

$$\begin{aligned}\psi^+_{-m+\frac12}\psi^+_{-m+\frac32}\cdots\psi^+_{-\frac32}\psi^+_{-\frac12}\cdot 1 &= q^m\\ \psi^-_{-m+\frac12}\psi^-_{-m+\frac32}\cdots\psi^-_{-\frac32}\psi^-_{-\frac12}\cdot 1 &= q^{-m}\end{aligned} \tag{2.3}$$

and

$$\psi_k^+\cdot q^m=0 \qquad \text{for}\quad k>-m \qquad \psi_k^-\cdot q^m=0 \qquad \text{for}\quad k>m. \tag{2.4}$$

Consider the vector space $V=V^+\oplus V^-$, where $V^{\pm}=\bigoplus_{i\in\mathbb{Z}+\frac12}\mathbb{C}\psi_i^{\pm}$ with symmetric bilinear form

$$\left(\psi_i^{\pm},\psi_j^{\pm}\right)=0 \qquad \left(\psi_i^{\pm},\psi_j^{\mp}\right)=\delta_{i,-j}. \tag{2.5}$$

We have thus constructed F as the spin module of the infinite Clifford algebra $C\ell V$ with relations ($u,v\in V$):

$$uv+vu=(u,v)1. \tag{2.6}$$

Define the bosonic fields

$$\alpha(z)=\sum_{k\in\mathbb{Z}}\alpha_k z^{-k-1}=:\psi^+(z)\psi^-(z): \tag{2.7}$$

where the normally ordered product between these fermions is defined as usual

$$:a_kb_\ell:=\begin{cases}a_kb_\ell & \text{if}\quad k\leqslant\ell\\ -b_\ell a_k & \text{if}\quad k>\ell.\end{cases}$$

The α_k form the oscillator algebra,

$$[\alpha_k,\alpha_\ell]=k\delta_{k,-\ell}$$

and

$$\begin{aligned}&\alpha_{-k}\cdot q^m=kt_kq^m\\ &\alpha_k\cdot q^m=0 \qquad \text{for}\quad k>0\\ &\alpha_0\cdot q^m=mq^m.\end{aligned}$$

In fact

$$\alpha_{-k}=kt_k \qquad \alpha_k=\frac{\partial}{\partial t_k} \qquad \alpha_0=q\frac{\partial}{\partial q}. \tag{2.8}$$

Consider the operator $\sigma=(-1)^{q\frac{\partial}{\partial q}}$. The Clifford algebra $C\ell V$ and the Fock space F decomposes into eigenspaces with respect to σ, i.e. $C\ell_{\overline{0}}V$ and $C\ell_{\overline{1}}V$ ($F_{\overline{0}}$ and $F_{\overline{1}}$) be the 1, respectively -1 eigenspaces, then $F_{\overline{\nu}}=\bigoplus_{k\in\nu+2\mathbb{Z}}F_k$.

Let $(C\ell V)^\times$ denote the multiplicative group of invertible elements of the algebra $C\ell V$. We denote by $\operatorname{Pin}V$ the subgroup of $(C\ell V)^\times$ generated by all the elements a such that $aVa^{-1}=V$ and let $\operatorname{Spin}V=\operatorname{Pin}V\cap C\ell_{\overline{0}}V$. Then clearly $F_{\overline{\nu}}$ is a $\operatorname{Spin}V$-module. Let $O_{\overline{0}}=\operatorname{Spin}V\cdot 1$ (resp. $O_{\overline{1}}=\operatorname{Spin}V\cdot q$) be the spin V-orbit of 1 (resp. q). Define the annihilator space $\operatorname{Ann}\tau$ for $\tau\in O_{\overline{\nu}}$:

$$\operatorname{Ann}\tau=\{v\in V|v\tau=0\}.$$

Then

$$\operatorname{Ann}1=\bigoplus_{k>0}\left(\mathbb{C}\psi_k^+\oplus\mathbb{C}\psi_k^-\right) \qquad \operatorname{Ann}q=\bigoplus_{k>-1}\mathbb{C}\psi_k^+\oplus\bigoplus_{k>1}\mathbb{C}\psi_k^-$$

are both maximal isotropic subspaces of V. Recall that a subspace $W \subset V$ is isotropic if $(v, w) = 0$ for any $v, w \in W$. An element $v \in V$ is called isotropic if $(v, v) = 0$ and anisotropic if $(v, v) \neq 0$. Since for $g \in \operatorname{Spin} V$

$$\operatorname{Ann} gq^{\nu} = \{gvg^{-1} | v \in \operatorname{Ann} q^{\nu}\}$$

we find that $\operatorname{Ann} \tau$ is a maximal isotropic subspace of V. In fact one can show (see [17]) that for any maximal isotropic subspace W of V such that W contains the subspace

$$\bigoplus_{k>N} (\mathbb{C}\psi_k^+ \oplus \mathbb{C}\psi_k^-)$$

for some $N \gg 0$, there exists a $g \in \operatorname{Spin} V$ such that either $W = \operatorname{Ann} g \cdot 1$ or $W = \operatorname{Ann} g \cdot q$. We call the collection of all such maximal isotropic subspaces of V the DKP Grassmannian. These group orbits are characterized by the following theorem [14, 17].

Theorem 2.1.

(a) If $\tau \in F_{\overline{\nu}}$ ($\nu \in \mathbb{Z}/2\mathbb{Z}$) and $\tau \neq 0$, then $\tau \in O_{\overline{\nu}}$ if and only if τ satisfies the (charged) DKP-equation or coupled KP equation:

$$\operatorname{Res}_z \psi^+(z)\tau \otimes \psi^-(z)\tau + \psi^-(z)\tau \otimes \psi^+(z)\tau = 0. \tag{2.9}$$

(b) Elements $\tau_\nu \in O_{\overline{\nu}}$, $\tau_\mu \in O_{\overline{\mu}}$, $\mu \neq \nu$, satisfy the Modified DKP hierarchy:

$$\operatorname{Res}_z \psi^+(z)\tau_\mu \otimes \psi^-(z)\tau_\nu + \psi^-(z)\tau_\mu \otimes \psi^+(z)\tau_\nu = \tau_\nu \otimes \tau_\mu \tag{2.10}$$

if and only if the space

$$(\operatorname{Ann} \tau_0 + \operatorname{Ann} \tau_1)/(\operatorname{Ann} \tau_0 \cap \operatorname{Ann} \tau_1)$$

is two-dimensional and the induced bilinear form on it is non-degenerate.

It is clear that any $\tau \in O_{\overline{\nu}}$ can be written as

$$\tau = \sum_{n \in \mathbb{Z}} \tau_n(t) q^n$$

where

$$\tau_n(t) = 0 \qquad \text{if} \quad n \notin \nu + 2\mathbb{Z}.$$

Substituting this and the vertex operators (2.1) into (2.9) and writing t' respectively t'' for the first and second components of the tensor product, we obtain the following hierarchy of differential equations. We find for all $n, m \in \mathbb{Z}$ with $m + n \in 2\mathbb{Z}$ the following equation:

$$\begin{aligned}\operatorname{Res}_z\, & z^{n-1} \mathrm{e}^{\xi(t',z)} \mathrm{e}^{-\eta(t',z)} \tau_{n-1}(t') z^{-m-1} \mathrm{e}^{-\xi(t'',z)} \mathrm{e}^{\eta(t'',z)} \tau_{m+1}(t'') \\ & + z^{-n-1} \mathrm{e}^{-\xi(t',z)} \mathrm{e}^{\eta(t',z)} \tau_{n+1}(t') z^{m-1} \mathrm{e}^{\xi(t'',z)} \mathrm{e}^{-\eta(t'',z)} \tau_{m-1}(t'') = 0.\end{aligned} \tag{2.11}$$

Make the change of variables

$$t_k = \tfrac{1}{2}(t_k' + t_k'') \qquad s_k = \tfrac{1}{2}(t_k' - t_k'') \tag{2.12}$$

and use the elementary Schur functions defined by

$$\sum_{k \in \mathbb{Z}} P_k(t) z^k = \mathrm{e}^{\xi(t,z)} \tag{2.13}$$

then (2.11) is equivalent to

$$\begin{aligned}\sum_{j=0}^{\infty} & P_j(2s) P_{j+n-m-1}\left(-\frac{\tilde{\partial}}{\partial s}\right) \tau_{n-1}(s+t)\tau_{m+1}(t-s) \\ & + P_j(-2s) P_{j+m-n-1}\left(\frac{\tilde{\partial}}{\partial s}\right) \tau_{n+1}(s+t)\tau_{m-1}(t-s) = 0.\end{aligned} \tag{2.14}$$

Here $\frac{\tilde{\partial}}{\partial s}$ stands for $\left(\frac{\partial}{\partial s_1}, \frac{1}{2}\frac{\partial}{\partial s_2}, \frac{1}{3}\frac{\partial}{\partial s_3}, \ldots\right)$. Using Taylor's formula this turns into the following generating series of Hirota bilinear equations:

$$\sum_{j=0}^{\infty} P_j(2s)P_{j+n-m-1}(-\tilde{D})\,\mathrm{e}^{\sum_{r=1}^{\infty} s_r D_r}\tau_{n-1}\cdot\tau_{m+1} + P_j(-2s)P_{j+m-n-1}(\tilde{D})\,\mathrm{e}^{\sum_{r=1}^{\infty} s_r D_r}\tau_{n+1}\cdot\tau_{m-1} = 0. \tag{2.15}$$

The simplest Hirota bilinear equations of the charged DKP hierarchy, appear in the constant term when we take $m-n=4$ ($m-n=0,2$ give trivial equations):

$$\left(D_1^3 + 3D_1D_2 + 2D_3\right)\tau_{n-2}\cdot\tau_n = 0. \tag{2.16}$$

The coefficient of s_3 with $n-m=2$ gives

$$\left(D_1^4 - 4D_1D_3 + 3D_2^2\right)\tau_n\cdot\tau_n = 24\tau_{n+2}\tau_{n-2}. \tag{2.17}$$

Now, keeping $n-m=2$ and taking the coefficient of s_4, s_5 and s_2s_3 gives all the other equations of (1.1). If we have $m-n=4$ and take the coefficients of s_1 and s_1^2 we obtain the second and last equations of (1.2). The coefficient of s_2 gives

$$\left(D_1^5 + 20D_1^2D_3 + 30D_1D_4 + 24D_5 - 15D_1D_2^2\right)\tau_{n-2}\cdot\tau_n.$$

Combining this with the last equation of (1.2) gives the third equation of (1.2). Finally taking $n-m=4$ and determining the coefficients of 1, s_1, s_1^2 and s_2 gives the Hirota bilinear equations of (1.3).

Remark 2.1. The Hirota bilinear equation (2.17) says the following. If we know two nonzero neighbours τ_n and τ_{n+2}, then using (2.17) we can determine all other τ_m and hence the whole τ.

3. Sato and Lax equations

We return to equation (2.11), or rather to the following equation which is equivalent to (2.11) if $n+m\in 2\mathbb{Z}$ and to the modified DKP hierarchy if $n+m\notin 2\mathbb{Z}$:

$$\begin{aligned}\mathrm{Res}_z\, & z^{n-1}\,\mathrm{e}^{\xi(t',z)}\,\mathrm{e}^{-\eta(t',z)}\tau_{n-1}(t')z^{-m-1}\,\mathrm{e}^{-\xi(t'',z)}\,\mathrm{e}^{\eta(t'',z)}\tau_{m+1}(t'') \\ & - (-z)^{-n-1}\,\mathrm{e}^{-\xi(t',-z)}\,\mathrm{e}^{\eta(t',-z)}\tau_{n+1}(t')(-z)^{m-1}\,\mathrm{e}^{\xi(t'',-z)}\,\mathrm{e}^{-\eta(t'',-z)}\tau_{m-1}(t'') \\ & = \tfrac{1}{2}(1-(-1)^{n+m})\tau_n(t')\tau_m(t'').\end{aligned} \tag{3.1}$$

We want to see this equation as one entry of a 2×2 matrix bilinear equation. Let

$$\begin{aligned} Q^{\pm}(t,z) &= \mathrm{diag}(\mathrm{e}^{\pm\xi(t,z)}, \mathrm{e}^{\mp\xi(t,-z)}) \\ R^{\pm}(n,\pm z) &= \mathrm{diag}(z^{\pm n}, (-z)^{\mp n}) \end{aligned} \tag{3.2}$$

$$P^{\pm}(n,t,\pm z) = \frac{1}{\tau_n(t)}\begin{pmatrix} \mathrm{e}^{\mp\eta(t,z)}\tau_n(t) & \mathrm{i}z^{-2}\,\mathrm{e}^{\pm\eta(t,-z)}\tau_{n\pm 2}(t) \\ -\mathrm{i}z^{-2}\,\mathrm{e}^{\mp\eta(t,z)}\tau_{n\mp 2}(t) & \mathrm{e}^{\pm\eta(t,-z)}\tau_n(t)\end{pmatrix} \tag{3.3}$$

and put

$$\Psi^{\pm}(n,t,z) = P^{\pm}(n,t,\pm z)R^{\pm}(n,\pm z)Q^{\pm}(t,z) \tag{3.4}$$

then (3.1) is equivalent to

$$\mathrm{Res}_z\,\Psi^{+}(n,t',z)^t\Psi^{-}(m,t'',z) = \frac{1-(-1)^{n+m}}{2\tau_n(t')\tau_m(t'')}\begin{pmatrix}\tau_{n+1}(t')\tau_{m-1}(t'') & -\mathrm{i}\tau_{n+1}(t')\tau_{m+1}(t'') \\ -\mathrm{i}\tau_{n-1}(t')\tau_{m-1}(t'') & -\tau_{n-1}(t')\tau_{m+1}(t'')\end{pmatrix}. \tag{3.5}$$

We call $\Psi^+(n,t,z)$ the nth wavefunction and $\Psi^-(n,t,z)$ the adjoint nth wavefunction. Writing x for t_1 and ∂ for ∂_x, we can express the wavefunctions as pseudo-differential operators (see [16] for more information) $P^\pm(n,t,\partial)R^\pm(n,\partial)$ acting on $Q^\pm(t,z)$

$$\Psi^\pm(n,t,z) = P^\pm(n,t,\partial)R^\pm(n,\partial)Q^\pm(t,z).$$

Note first that

$$R^-(n,\partial) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} R^+(n,\partial) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$P^-(n,t,\partial) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P^+(n,t,\partial) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Using standard techniques, see e.g. [16] we obtain

$$(P^+(n,t,\partial)R^+(n,\partial)R^-(m,\partial)^*P^-(m,t,\partial)^*)_- = \frac{1-(-1)^{n+m}}{2}\begin{pmatrix} \frac{\tau_{n+1}(t)}{\tau_n(t)}\partial^{-1}\frac{\tau_{m-1}(t)}{\tau_m(t)} & -\mathrm{i}\frac{\tau_{n+1}(t)}{\tau_n(t)}\partial^{-1}\frac{\tau_{m+1}(t)}{\tau_m(t)} \\ -\mathrm{i}\frac{\tau_{n-1}(t)}{\tau_n(t)}\partial^{-1}\frac{\tau_{m-1}(t)}{\tau_m(t)} & -\frac{\tau_{n-1}(t)}{\tau_n(t)}\partial^{-1}\frac{\tau_{m+1}(t)}{\tau_m(t)} \end{pmatrix}. \tag{3.6}$$

Since $R^-(m,\partial)^* = R^+(-m,\partial) = R^+(m,\partial)^{-1}$, we deduce from (3.6) for $m = n$ that $P^-(m,t,\partial)^* = P^+(m,t,\partial)^{-1}$, hence (3.6) is equivalent to

$$(P^+(n,t,\partial)R^+(n-m,\partial)P^+(m,t,\partial)^{-1})_- = \frac{1-(-1)^{n+m}}{2}\begin{pmatrix} \frac{\tau_{n+1}(t)}{\tau_n(t)}\partial^{-1}\frac{\tau_{m-1}(t)}{\tau_m(t)} & -\mathrm{i}\frac{\tau_{n+1}(t)}{\tau_n(t)}\partial^{-1}\frac{\tau_{m+1}(t)}{\tau_m(t)} \\ -\mathrm{i}\frac{\tau_{n-1}(t)}{\tau_n(t)}\partial^{-1}\frac{\tau_{m-1}(t)}{\tau_m(t)} & -\frac{\tau_{n-1}(t)}{\tau_n(t)}\partial^{-1}\frac{\tau_{m+1}(t)}{\tau_m(t)} \end{pmatrix}. \tag{3.7}$$

Next take $n = m$ in (3.5) and differentiate to t'_j, then one obtains (again using standard techniques) the Sato equation

$$\frac{\partial P^+(n,t,\partial)}{\partial t_j} = -\left(P^+(n,t,\partial)\begin{pmatrix} \partial^j & 0 \\ 0 & -(-\partial)^j \end{pmatrix} P^+(n,t,\partial)^{-1}\right)_- P^+(n,t,\partial). \tag{3.8}$$

Introduce the pseudo-differential operators

$$L(n,t,\partial) = P^+(n,t,\partial)\begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix} P^+(n,t,\partial)^{-1}$$

$$J(n,t,\partial) = P^+(n,t,\partial)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P^+(n,t,\partial)^{-1}$$

then it is straightforward to show that $[L(n,t,\partial), J(n,t,\partial)] = 0$ and that the following Lax equations hold:

$$\begin{aligned} \frac{\partial L(n,t,\partial)}{\partial t_j} &= [(L(n,t,\partial)^j J(n,t,\partial))_+, \partial L(n,t,\partial)] \\ \frac{\partial J(n,t,\partial)}{\partial t_j} &= [(L(n,t,\partial)^j J(n,t,\partial))_+, \partial J(n,t,\partial)]. \end{aligned} \tag{3.9}$$

The Sato equation leads to the following linear system:

$$\begin{aligned} L(n,t,\partial)\Psi^+(n,t,z) &= \Psi^+(n,t,z)\begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \\ J(n,t,\partial)\Psi^+(n,t,z) &= \Psi^+(n,t,z)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \frac{\partial \Psi^+(n,t,z)}{\partial t_j} &= (L(n,t,\partial)^j J(n,t,\partial))_+\Psi^+(n,t,z). \end{aligned} \tag{3.10}$$

Using (3.5) one can show, in a similar way as in [16], that there exist differential operators $D(n \pm 2, n, t, \partial)$, which are completely determined by $P^+(n, t, \partial)$ such that

$$\Psi^+(n \pm 2, t, z) = D(n \pm 2, n, t, \partial)\Psi^+(n, t, z).$$

Since we do not need this in the rest of the paper we shall not give the explicit form of the $D(n \pm 2, n, t, \partial)$ here. Note that this reflects the statement of remark 2.1 for wavefunctions.

In the next section we will need the notion of 'DKP or coupled KP eigenvectors' generalizing the notion of KP eigenfunctions for the KP hierarchy. We call the vectors

$$\Phi(n, t) = \begin{pmatrix} \phi_1(n, t) \\ \phi_2(n, t) \end{pmatrix}$$

coupled KP eigenvectors if they satisfy

$$\frac{\partial \Phi^+(n, t)}{\partial t_j} = (L(n, t, \partial)^j J(n, t, \partial))_+ \Phi(n, t). \tag{3.11}$$

One can construct such coupled KP eigenvectors from the wavefunction $\Psi^+(n, t, z)$ as follows. Let $a^i(z) = \sum_{j \in \mathbb{Z}} a^i_j z^j$, then

$$\Phi(n, t) = \mathrm{Res}_z\, \Psi^+(n, t, z) \begin{pmatrix} a^1(z) \\ a^2(z) \end{pmatrix}.$$

Kakei [19] obtained this coupled KP hierarchy in a different way, namely as a reduction of the two-component KP hierarchy.

4. Bäcklund–Darboux transformations

Elementary Bäcklund–Darboux transformations are based on the following simple observations in the Clifford algebra and Spin group orbit. First, if $\alpha \in V$ is a anisotropic vector, i.e. $(\alpha, \alpha) \neq 0$, then, by (2.6)

$$\alpha^{-1} = \frac{2\alpha}{(\alpha, \alpha)}. \tag{4.1}$$

Hence $\alpha \in (C\ell V)^{\times}$. From (2.6) and (4.1) we obtain

$$\alpha v \alpha^{-1} = -v + 2\frac{(v, \alpha)}{(\alpha, \alpha)}\alpha = -r_\alpha(v) \tag{4.2}$$

thus even $\alpha \in \mathrm{Pin}\, V$.

Note next that

$$q = \psi^+_{-\frac{1}{2}} \cdot 1 = \left(\psi^+_{-\frac{1}{2}} + \psi^-_{\frac{1}{2}}\right) \cdot 1 \qquad 1 = \tfrac{1}{2}\left(\psi^+_{-\frac{1}{2}} + \psi^-_{\frac{1}{2}}\right) \cdot q.$$

Let $\tau \in O_{\overline{\nu}}$, then we can write τ as $g \cdot 1$ (or $g \cdot q$) for $\overline{\nu} = \overline{0}$ (respectively $\overline{\nu} = \overline{1}$) for certain $g \in \mathrm{Spin}\, V$. Take an $\alpha \in V$, then α can be anisotropic or isotropic. If α is anisotropic then $\alpha g \in \mathrm{Pin}\, V$ and

$$\alpha g \left(\psi^+_{-\frac{1}{2}} + \psi^-_{\frac{1}{2}}\right) \in \mathrm{Spin}\, V.$$

Thus $\alpha\tau \in O_{\overline{\nu+1}}$ or $\alpha\tau = 0$. If α is isotropic and $\alpha\tau \neq 0$, we can find a $\beta \in \mathrm{Ann}\,\tau$ such that $(\alpha, \beta) \neq 0$. Then $\alpha + \beta$ is anisotropic and $\alpha\tau = (\alpha + \beta)\tau$, thus again $\alpha\tau \in O_{\overline{\nu+1}}$.

Assume now, without loss of generality that α is anisotropic, then $\mathrm{Ann}\,\alpha\tau = r_\alpha\,(\mathrm{Ann}\,\tau)$. Let V_α consist of all the elements $v \in V$ such that $(v, \alpha) = 0$. Then $r_\alpha(v) = v$ for $v \in V_\alpha$ and $V_\alpha \subset V$ of codimension 1. Hence $\mathrm{Ann}\,\tau \cap V_\alpha \subset \mathrm{Ann}\,\tau$ of codimension 0 or 1. Since there exists a $w \in \mathrm{Ann}\,\tau$ such that $(w, r_\alpha(w)) \neq 0$, $r_\alpha(w) \notin \mathrm{Ann}\,\tau$ and similarly $w \notin r_\alpha(\mathrm{Ann}\,\tau)$.

Hence τ and $\alpha\tau$ satisfy the conditions of theorem 2.1. Hence we have proved the following important proposition.

Proposition 4.1. *If $\tau \in O_{\overline{\nu}}$ and $\alpha \in V$, then $\alpha\tau \in O_{\overline{\nu+1}} \cup \{0\}$ and if moreover $\alpha\tau \neq 0$, then*

$$(\mathrm{Ann}\,\tau + \mathrm{Ann}\,\alpha\tau)/(\mathrm{Ann}\,\tau \cap \mathrm{Ann}\,\alpha\tau)$$

is two-dimensional and the induced bilinear form on it is non-degenerate.

In other words, if τ satisfies the DKP hierarchy then $\alpha\tau$ also satisfies the DKP hierarchy and both τ and $\alpha\tau$ satisfy the modified DKP hierarchy. This observation is basic and forms the basis of the elementary Bäcklund–Darboux transformations.

Remark 4.1. From section 2 we know that to any (algebraic) solution τ of the coupled KP hierarchy there corresponds a maximal isotropic subspace $\mathrm{Ann}\,\tau$ of V. By taking any non-isotropic vector $\alpha \in V$, we obtain a new solution of the coupled KP, whose maximal isotropic subspace is $r_\alpha(\mathrm{Ann}\,\tau)$. Hence, elementary Bäcklund–Darboux transformations are related to simple reflections

$$r_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \qquad v \in V$$

in the orthogonal group corresponding to V. The coupled KP solution τ and the choice of $\alpha \in V$ will determine the form of the elementary Bäcklund–Darboux transformation uniquely. This of course up to a scalar factor, since any scalar multiple of τ is again a solution.

In fact, if we write such an anisotropic $\alpha \in V$ as $\alpha = \beta + \gamma$ with $(\beta,\beta) = (\gamma,\gamma) = 0$ and $\beta \in \mathrm{Ann}\,\tau$ and $\gamma \notin \mathrm{Ann}\,\tau$, then $\beta \notin r_\alpha(\mathrm{Ann}\,\tau)$ and $\gamma \in r_\alpha(\mathrm{Ann}\,\tau)$ and

$$\begin{aligned} r_\alpha(\mathrm{Ann}\,\tau) \cap \mathrm{Ann}\,\tau &= \{v \in \mathrm{Ann}\,\tau | (v,\gamma) = 0\} \\ &= \{v \in r_\alpha(\mathrm{Ann}\,\tau) | (v,\beta) = 0\}. \end{aligned}$$

We will now describe the elementary Bäcklund–Darboux transformations explicitly. Every element $\alpha \in V$ (which may be isotropic) can be written as a linear combination of the basis vectors $\psi_j^\pm$. So let

$$\alpha = \sum_{j \in \frac{1}{2}+\mathbb{Z}} c_{-j}^+ \psi_j^+ + c_{-j}^- \psi_j^- \tag{4.3}$$

then we can rewrite α as

$$\alpha = \mathrm{Res}_z\, c^+(z)\psi^+(z) - c^-(-z)\psi^-(-z)$$

where

$$c^\pm(z) = \sum_{j \in \frac{1}{2}+\mathbb{Z}} c_j^\pm z^{-z-\frac{1}{2}}.$$

Let $\tau \in O_{\overline{0}}$, then we can write $\tau = \sum_n \tau_n(t) q^n$, with $\tau_n(t) = 0$ for $n \bmod 2 \neq \overline{0}$. Write $\sigma = \alpha\tau$, with $\sigma = \sum_n \sigma_n q^n$ Then clearly,

$$\begin{aligned} \sigma &= \sum_n \mathrm{Res}_z (c^+(z)\psi^+(z) - c^-(-z)\psi^-(-z))\tau_n(t) q^n \\ &= \mathrm{Res}_z\, c^+(z) z^n\, \mathrm{e}^{-\eta(t,z)} \tau_n(t)\, \mathrm{e}^{\xi(t,z)} q^{n+1} - c^-(-z) z^{-n}\, \mathrm{e}^{\eta(t,-z)} \tau_n(t)\, \mathrm{e}^{-\xi(t,-z)} q^{n-1}. \end{aligned}$$

Hence,

$$\sigma_m(t) = \mathrm{Res}_z\, c^+(z) z^{m-1}\, \mathrm{e}^{-\eta(t,z)} \tau_{m-1}(t)\, \mathrm{e}^{\xi(t,z)} - c^-(-z) z^{-m-1}\, \mathrm{e}^{\eta(t,-z)} \tau_{m+1}(t)\, \mathrm{e}^{-\xi(t,-z)}.$$

So, if we introduce the coupled KP eigenvectors

$$\Phi(\alpha; m, t) = \begin{pmatrix} \phi_1(\alpha; m, t) \\ \phi_2(\alpha; m, t) \end{pmatrix} = \mathrm{Res}_z\, \Psi^+(m, t, z) \begin{pmatrix} c^+(z) \\ \mathrm{i}c^-(-z) \end{pmatrix}$$

then

$$\begin{pmatrix} \sigma_m(t) \\ -\mathrm{i}\sigma_{m-2}(t) \end{pmatrix} = \tau_{m-1}(t)\Phi(\alpha; m-1, t). \tag{4.4}$$

We next want to determine the wavefunction which corresponds to σ. Assume $\tau_n(t) \neq 0$ then $\Psi^+(n, t, z)$ is a wavefunction that corresponds to τ and consider $\Psi^+(n \pm 1, t, z)$ to correspond to σ. Since σ and τ satisfy the DKP hierarchy $P^+(n, t, \partial)$ and $P^+(n \pm 1, t, \partial)$ satisfy (3.7). Then it is straightforward to show that

$$\begin{aligned} &P^+(n+1, t, \partial)R^+(1, \partial)P^+(n, t, \partial)^{-1} \\ &\quad = \begin{pmatrix} \frac{\tau_{n+1}(t)}{\tau_n(t)}\partial\frac{\tau_n(t)}{\tau_{n+1}(t)} + \frac{\tau_{n+2}(t)}{\tau_{n+1}(t)}\partial^{-1}\frac{\tau_{n-1}(t)}{\tau_n(t)} & -\mathrm{i}\frac{\tau_{n+2}(t)}{\tau_{n+1}(t)}\partial^{-1}\frac{\tau_{n+1}(t)}{\tau_n(t)} \\ -\mathrm{i}\frac{\tau_n(t)}{\tau_{n+1}(t)}\partial^{-1}\frac{\tau_{n-1}(t)}{\tau_n(t)} & -\frac{\tau_n(t)}{\tau_{n+1}(t)}\partial^{-1}\frac{\tau_{n+1}(t)}{\tau_n(t)} \end{pmatrix} \\ &P^+(n-1, t, \partial)R^+(1, \partial)P^+(n, t, \partial)^{-1} \\ &\quad = \begin{pmatrix} \frac{\tau_n(t)}{\tau_{n-1}(t)}\partial^{-1}\frac{\tau_{n-1}(t)}{\tau_n(t)} & -\mathrm{i}\frac{\tau_n(t)}{\tau_{n-1}(t)}\partial^{-1}\frac{\tau_{n+1}(t)}{\tau_n(t)} \\ -\mathrm{i}\frac{\tau_{n-2}(t)}{\tau_{n-1}(t)}\partial^{-1}\frac{\tau_{n-1}(t)}{\tau_n(t)} & -\frac{\tau_{n-1}(t)}{\tau_n(t)}\partial\frac{\tau_n(t)}{\tau_{n-1}(t)} - \frac{\tau_{n-2}(t)}{\tau_{n-1}(t)}\partial^{-1}\frac{\tau_{n+1}(t)}{\tau_n(t)} \end{pmatrix}. \end{aligned} \tag{4.5}$$

Now using (4.4), we can rewrite this into

$$\begin{aligned} P^+(n+1, t, \partial)R^+(1, \partial)P^+(n, t, \partial)^{-1} &= D(\alpha; n+1, n, t, \partial) \\ P^+(n-1, t, \partial)R^+(1, \partial)P^+(n, t, \partial)^{-1} &= D(\alpha; n-1, n, t, \partial) \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} D(\alpha; n+1, n, t, \partial) &= \begin{pmatrix} \phi_1(\alpha; n, t) \\ 0 \end{pmatrix} \partial (\phi_1(\alpha; n, t)^{-1} \quad 0) \\ &\quad + \begin{pmatrix} \phi_2(\alpha; n+2, t)^{-1} \\ -\phi_1(\alpha; n, t)^{-1} \end{pmatrix} \partial^{-1} (\phi_2(\alpha; n, t) \quad \phi_1(\alpha; n, t)) \\ D(\alpha; n-1, n, t, \partial) &= \begin{pmatrix} 0 \\ -\phi_2(\alpha; n, t) \end{pmatrix} \partial (0 \quad \phi_2(\alpha; n, t)^{-1}) \\ &\quad + \begin{pmatrix} \phi_2(\alpha; n, t)^{-1} \\ -\phi_1(\alpha; n-2, t)^{-1} \end{pmatrix} \partial^{-1} (\phi_2(\alpha; n, t) \quad \phi_1(\alpha; n, t)) \end{aligned} \tag{4.7}$$

are so-called elementary Bäcklund–Darboux operators for the coupled KP hierarchy, i.e

$$\Psi^+(n \pm 1, t, z) = D(\alpha; n \pm 1, n, t, \partial)\Psi^+(n, t, z).$$

We formulate the above results in the main theorem of this paper, namely.

Theorem 4.1. *Suppose that $\tau = \sum_n \tau_n(t)q^n \in O_{\overline{\nu}}$ and let $\Psi^+(n, t, z)$ for $n \in \overline{\nu} + 2\mathbb{Z}$ be the corresponding wavefunction. Let*

$$\alpha = \mathrm{Res}_z\, c^+(z)\psi^+(z) - c^-(-z)\psi^-(-z) \in V$$

where $c^\pm(z) \in \mathbb{C}[z, z^{-1}]$, then $\sigma = \sum_n \sigma_n(t)q^n = \alpha\tau \in O_{\overline{\nu+1}}$ or $= 0$ and both σ and τ satisfy the modified DKP hierarchy. Let

$$\Phi(\alpha; n, t) = \begin{pmatrix} \phi_1(\alpha; n, t) \\ \phi_2(\alpha; n, t) \end{pmatrix} = \mathrm{Res}_z\, \Psi^+(n, t, z) \begin{pmatrix} c^+(z) \\ \mathrm{i}c^-(-z) \end{pmatrix}$$

for $n \in \overline{\nu} + 2\mathbb{Z}$ *be the to* α *and* τ *corresponding eigenfunctions, then*

$$\begin{pmatrix} \sigma_{n+1}(t) \\ -\mathrm{i}\sigma_{n-1}(t) \end{pmatrix} = \tau_n(t)\Phi(\alpha; n, t).$$

Write $\Psi^+(n \pm 1, t, z)$ *for the to* σ *corresponding wavefunctions, then*

$$\Psi^+(n \pm 1, t, z) = D(\alpha; n \pm 1, n, t, \partial)\Psi^+(n, t, z)$$

where the $D(\alpha; n\pm 1, n, t, \partial)$ *are the elementary Bäcklund–Darboux operators for the coupled KP hierarchy as defined in (4.7).*

Moreover, if α *is anisotropic then*

$$\mathrm{Ann}\,\sigma = r_\alpha(\mathrm{Ann}\,\tau).$$

Now starting with the simplest solution of this coupled KP hierarchy, namely $\tau_0 = 1$ and all other $\tau_n = 0$ for $n \neq 0$. This is the solution related to $1 \in F$. One has as wavefunction

$$\psi^{\pm}(0, t, \pm z) = \begin{pmatrix} \mathrm{e}^{\mp\eta(t,z)} & 0 \\ 0 & \mathrm{e}^{\pm\eta(t,z)} \end{pmatrix}.$$

Now repeatedly applying elementary Bäcklund–Darboux transformations with various Laurent polynomials $c^{\pm}(z)'s$, one obtains more complicated solutions. If one replaces the Laurent polynomials by certain infinite series e.g. some combination of delta functions $\delta(z - a) = z^{-1}\sum_{n\in\mathbb{Z}} \left(\frac{z}{a}\right)^n$, one leaves the algebraic framework of this paper, however, one still obtains solutions of the coupled KP.

If we start again from $\tau = 1$ and we choose at every step of this process one of the $c^+(z)$ or $c^-(z)$ always equal to zero, we obtain tau-functions that satisfy the KP hierarchy.

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